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CHANG'S CONJECTURE FOR $\aleph_\omega$

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ABSTRACT

We establish, starting from some assumptions of the order of magnitude of a huge cardinal, the consistency of $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$, as well as of some other transfer properties of the type $(\kappa^+, \kappa) \twoheadrightarrow (\alpha^+, \alpha)$, where κ is singular.

0. Notations

0.1. On denotes the class of all ordinals.
$|X|$ denotes the cardinality of X.

0.2. If λ is a cardinal and α is an ordinal, $\lambda^{+\alpha}$ denotes the αth successor of λ.

0.3. If μ is a cardinal, H_μ is the set of all sets of hereditary cardinality $<\mu$.

0.4. If p is a function, dom(p) is the domain of p and Rg(p) is the range of p.

0.5. Let A be a structure. dom(A) is the base set of A. "$X \prec A$" means that X is an elementary substructure of A. Hull(X, A) is the Skolem Hull of X in A (with respect to a fixed well-order of dom(A)).

0.6. If j is an elementary embedding, cp(j) is the critical point of j.

0.7. If G is generic over V and $a \in V$, a_G denotes the G-interpretation of a in $V[G]$.

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1. Chang's Conjectures

1.1. Let L be a first-order language. Let $\boldsymbol{A}^*$, $\boldsymbol{B}^*$ be two classes of L-structures. Chang's Conjecture for $(\boldsymbol{A}^*, \boldsymbol{B}^*)$ (over L) can be stated as follows: for every $\boldsymbol{A} \in \boldsymbol{A}^*$ there exists $\boldsymbol{B} \in \boldsymbol{B}^*$ such that $\boldsymbol{B} \prec \boldsymbol{A}$. We usually write $\boldsymbol{A}^* \twoheadrightarrow \boldsymbol{B}^*$.

1.2. Usually, we consider a two-cardinal form of Chang's Conjecture. Let κ, λ, β, α be four infinite cardinals. We write $(\kappa, \lambda) \twoheadrightarrow (\beta, \alpha)$ iff the following holds: for every countable first-order language L and for every L-structure $A = (\kappa, \lambda, \ldots)$ there exists an $X \prec A$ such that $|X| = \beta$ and $|X \cap \lambda| = \alpha$.

1.3. Remarks. (1) It is clear that we obtain the same conjecture if we look only at languages L such that $|L| < \omega$, or even only at languages L such that $|L| \leqq 2$, as soon as L contains at least a binary predicate.

(2) We can also introduce a fifth cardinal μ and write $(\kappa, \lambda) \twoheadrightarrow_{<\mu} (\beta, \alpha)$ meaning that we have got to take care of all the languages L such that $|L| < \mu$.

(3) We can also define the symbol $(\kappa, \lambda) \twoheadrightarrow (\beta, <\alpha)$, which means that the elementary substructure X should be such that $|X| = \beta$ and $|X \cap \lambda| < \alpha$.

One can consult [R] for more information.

1.4. Let us go back to the case $(\kappa, \lambda) \twoheadrightarrow (\beta, \alpha)$. Several cases are "irrelevant", in the following sense: they are either provably false or provably true (in ZFC). In order to avoid these trivial cases, we must have: $\kappa > \lambda$, $\beta > \alpha$, $\kappa \geqq \beta$, $\lambda > \alpha$.

1.5. For instance, we can consider conjectures like

$$(\omega_5, \omega_3) \twoheadrightarrow (\omega_4, \omega_2)$$
$$(\omega_3, \omega_1)$$
$$(\omega_2, \omega_0)$$
$$(\omega_3, \omega_2).$$

1.6. Note that a finite gap can never be increased. For example, $(\omega_5, \omega_4) \twoheadrightarrow (\omega_3, \omega_1)$ is inconsistent, because we can code the fact that "the universe is $\leqq$ to the successor of ω_4".

1.7. More Examples. (1) The usual Conjecture of Chang is $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega_0)$.

(2) $(\kappa, \omega_1) \twoheadrightarrow (\kappa, \omega_0)$ means that κ is ω_1-Rowbottom.

(3) $(\forall\lambda<\kappa)[(\kappa,\lambda)\twoheadrightarrow(\kappa,\omega_0)]$ means that κ is Rowbottom.

(4) Set $A^*=\{A/\mathrm{dom}(A)=\kappa\}$ and

$$B^*=\{B/\mathrm{dom}(B)\subseteq\kappa,\ \mathrm{dom}(B)\neq\kappa,\ |\mathrm{dom}(B)|=\kappa\}.$$

$A^*\twoheadrightarrow B^*$ means that κ is Jonsson.

1.8. Consistency. (1) The consistency of $(\omega_2,\omega_1)\twoheadrightarrow(\omega_1,\omega_0)$ has been established early by Silver (unpublished), starting with an ω_1-Erdös cardinal.

(2) Later proofs have been given by Baumgartner (unpublished) and in [DL], without use of Martin's Axiom, and also using a Levy collapse instead of a Silver collapse.

(3) The consistency of $(\omega_{n+2},\omega_{n+1})\twoheadrightarrow(\omega_{n+1},\omega_n)$ for $n\geqq 1$ has been established in [K], starting with a huge cardinal.

The consistency of other forms of Chang's Conjecture can be found in [F].

1.9. Converses. As to converses, the following is known.

(1) [DJK] shows that, if $(\omega_2,\omega_1)\twoheadrightarrow(\omega_1,\omega_0)$ holds and if we set $\kappa=\omega_2$ and $\lambda=\omega_1$, then $K\models$ "κ is λ-Erdös" (where K denotes the Core Model).

(2) Indeed, the proof of [DJK] shows also the following. If, for some infinite cardinals κ and λ, $(\kappa^+,\kappa)\twoheadrightarrow(\lambda^+,\lambda)$ holds, then, setting $\mu=\lambda^+$, $K\models$ "there exists a μ-Erdös cardinal".

1.10. When we look at conjectures of the form $(\kappa^+,\kappa)\twoheadrightarrow(\lambda^+,\lambda)$, we have two quite different situations, depending on whether $\lambda=\omega_0$ or $\lambda\geqq\omega_1$.

(1) We know that, if $(\kappa^+,\kappa)\twoheadrightarrow(\omega_1,\omega_0)$, then $K\models$ "there exists an ω_1-Erdös cardinal".

(2) Conversely, if we start with an ω_1-Erdös cardinal, we can obtain the consistency of almost any conjecture of the form $(\kappa^+,\kappa)\twoheadrightarrow(\omega_1,\omega_0)$, as long as we accept κ to be regular (for example, κ can be ω_7, or the first inaccessible cardinal).

1.11. Higher Cardinals. The situation changes, if we look at $(\kappa^+,\kappa)\twoheadrightarrow(\lambda^+,\lambda)$, where $\lambda\geqq\omega_1$.

(1) It is proved in [DK] that, if $(\kappa^{++},\kappa^+)\twoheadrightarrow(\kappa^+,\kappa)$ and $\kappa\geqq\omega_1$, then $0^\dagger$ ["0 dagger"] exists.

(2) [L] derives the existence of $0^\dagger$ from other forms of Chang's Conjecture, such as the following:

(a) $(\kappa^+,\kappa)\twoheadrightarrow(\lambda^+,\lambda)$ and $\lambda\geqq\omega_1$,

(b) $(\kappa^{+n},\kappa)\twoheadrightarrow(\lambda^{+m},\lambda)$ and $m\geqq 2$,

(c) $(\kappa^+,\kappa)\twoheadrightarrow(\omega_1,\omega_0)$ and κ is singular.

One can also, for more on the existence of $0^\dagger$ following from some forms of Chang's Conjecture, look at [DKL].

1.12. Hence, by 1.11(2)(c), "$(\kappa^+,\kappa)\twoheadrightarrow(\omega_1,\omega_0)$ and κ is singular" is already a rather large cardinal axiom.

The rest of this paper is devoted to establishing the consistency of various hypotheses of the form $(\kappa^+,\kappa)\twoheadrightarrow(\lambda^+,\lambda)$, where κ is singular, in particular of $(\aleph_{\omega+1},\aleph_\omega)\twoheadrightarrow(\omega_1,\omega_0)$.

1.13. Before proceeding further, we may already note a fact, which puts some limits on the possibility of this kind of conjecture.

Lemma 1. *Assume that κ,ρ are infinite cardinals such that κ is singular and $\rho^{\mathrm{cof}(\kappa)}=\rho$ (in particular, $\mathrm{cof}(\kappa)<\rho$). Then: $(\kappa^+,\kappa)\not\twoheadrightarrow(\rho^+,\rho)$.*

Proof of Lemma 1. Assume that $(\kappa^+,\kappa)\twoheadrightarrow(\rho^+,\rho)$. Set $\theta=\mathrm{cof}(\kappa)$. Hence, $\theta<\rho$. Clearly, $\kappa^\theta\geqq\kappa^+$. Let $(f_i)_{i<\kappa^+}$ be a family of κ^+ different functions, $f_i:\theta\to\kappa$. Set $F=\bigcup_{i<\kappa^+}\{i\}\times f_i$ and $A=(\kappa^+,\theta,F)$. Assume that $X\prec A$, $|X|=\rho^+$ and $|X\cap\kappa|=\rho$. By elementarity, the family $f_i\restriction_X: X\cap\theta\to X\cap\kappa$ is made of $|X|$ different functions. But $|X\cap\kappa|^{|X\cap\theta|}\leqq\rho^\theta\leqq\rho$, a contradiction. QED Lemma 1.

2. Elementary submodels

2.1. Before dealing with singular cardinals, we shall recall a possible way of proving that certain forms of Chang's Conjecture are preserved in some generic extensions.

First of all, let us recall the following lemma.

Lemma 2. *Assume that $\kappa,\lambda,\beta,\alpha$ are four infinite cardinals. The following are equivalent.*

(1) $(\kappa,\lambda)\twoheadrightarrow(\beta,\alpha)$.

(2) *For at least one regular cardinal $\mu\geqq\kappa$, the following holds: for every language L with $|L|\leqq\omega$ and every L-structure $B=(H_\mu,\in,L^B)$, there exists $Y\prec B$ such that*
   (a) $|Y\cap\kappa|=\beta$,
   (b) $|Y\cap\lambda|=\alpha$.

(3) *For every regular cardinal $\mu\geqq\kappa$, (2) holds.*

Proof of Lemma 2. (3)→(2) is obvious.

(2)→(1). Assume that L is a countable language and that $A=(\kappa,\lambda,L^A)$ is an L-structure. Set $\boldsymbol{B} = (H_\mu, \in, \kappa, \lambda, L^A)$ and let $Y \prec \boldsymbol{B}$ be such that $|Y \cap \kappa| = \beta$ and $|Y \cap \lambda| = \alpha$. It is clear that $Y \cap \kappa \prec_L \boldsymbol{A}$.

(1)→(3). Let μ be regular, such that $\mu \geqq \kappa$. Let L be a countable language, and $\boldsymbol{B} = (H_\mu, \in, L^{\boldsymbol{B}}, \lhd)$ be a structure, where $\lhd$ is a well-order on H_μ. Let $(f_n)_{n<\omega}$, $f_n : [H_\mu]^n \to H_\mu$, be a complete set of Skolem functions for $\boldsymbol{B}$ and define, for $n < \omega$, a function $g_n : [\kappa]^n \to \kappa$, setting, for $a \in [\kappa]^n$,

$$g_n(a) = \begin{cases} f_n(a) & \text{if } f_n(a) \in \lambda, \\ 0 & \text{if not.} \end{cases}$$

Finally, set $\boldsymbol{A} = (\kappa, \lambda, (g_n)_{n<\omega})$. Let, by (1), X be such that $X \prec \boldsymbol{A}$, $|X| = \beta$ and $|X \cap \lambda| = \alpha$. Set $Y = \mathrm{Hull}(X, \boldsymbol{B})$. It is clear that $|Y| = |Y \cap \kappa| = |X| = \beta$. On the other hand, due to the g_n's, we have $|Y \cap \lambda| = \alpha$. QED Lemma 2.

2.2. Let us now, as a sample, recall the following.

THEOREM 3. *Assume that* $(\kappa, \alpha^+) \twoheadrightarrow (\alpha^+, \alpha)$. *Let* P *be a set of conditions, such that* P *satisfies the* α^+*-antichain condition. Then*:

$$V^P \models (\kappa, \alpha^+) \twoheadrightarrow (\alpha^+, |\alpha|).$$

PROOF OF THEOREM 3. Let μ be a regular cardinal, large enough so that $\mathscr{P}(P) \subseteq H_\mu$. Let G be P-generic over V. Since $\mathscr{P}(P) \subseteq H_\mu$, $(H_\mu)^{V[G]} = H_\mu[G]$. Let, in $V[G]$, $\bar{\boldsymbol{B}} = (H_\mu[G], \in, \bar{B})$ be a structure. Take $B \in V$, such that $B \subseteq H_\mu$ and $\bar{B} = (B)_G$. Set $\boldsymbol{B} = (H_\mu, \in, B, \{\alpha\}, \lhd)$, where $\lhd$ is a well-order of H_μ. Hence, $\bar{\boldsymbol{B}}$ is a reduct of $\boldsymbol{B}[G]$.

CLAIM 1. There exists, in V, an $M \prec \boldsymbol{B}$ such that $|M \cap \kappa| = \alpha^+$, $|M \cap \alpha^+| = \alpha$, $\alpha \in M$ and $M \cap \alpha^+$ is transitive.

PROOF OF CLAIM 1. Let us work for a moment in V. Let $X \prec \boldsymbol{B}$ be such that $|X| = \alpha^+$ and $|X \cap \alpha^+| = \alpha$. Set $\xi_0 = \sup(X \cap \alpha^+)$. Clearly, $\xi_0 < \alpha^+$. Hence, $\alpha < \xi_0 < \alpha^+$. Set $M = \mathrm{Hull}(X \cup \xi_0, \boldsymbol{B})$. It is clear that $|M| = \alpha^+$. Hence, it is enough to show that $M \cap \alpha^+ = \xi_0$. To see this, let $t(v_1, \ldots, v_n, w)$ be a term, and let $\delta_1, \ldots, \delta_n < \xi_0$ and $a \in X$ be such that $\zeta = t(\delta_1, \ldots, \delta_n, a) \in \alpha^+$. We have got to show that $\zeta < \xi_0$. Take $\eta \in X \cap \xi_0$ such that $\delta_1, \ldots, \delta_n < \eta$, and set $s(\eta, a) = \{t(\gamma_1, \ldots, \gamma_n, a) / \gamma_1, \ldots, \gamma_n < \eta\}$. Clearly, $s(\eta, a) \in X$, and $t(\vec{\delta}, a) \in s(\eta, a) \cap \alpha^+$. Hence, setting $\gamma = \sup[s(\eta, a) \cap \alpha^+]$ we see, since $\alpha^+ \in X$, that $t(\vec{\delta}, a) \leqq \gamma < \xi_0$. QED Claim 1.

Let M be as in Claim 1. Let $h : N \underset{\sim}{\to} M$ be the transitive collapse of M. Set $P' = (h^{-1})(P)$ and $H = (h^{-1})''(G)$.

CLAIM 2. H is P'-generic over N.

PROOF OF CLAIM 2. Let D be a maximal antichain in P', such that D is first-order definable in N. By elementarity, $D \in N$ [since $\boldsymbol{B}$ satisfies that every definable subset of P is a set]. By elementarity again, $N \models$ "$|D| \leqq \alpha$", since P satisfies the α^+-antichain condition in V. But $h \restriction_{\alpha+1} = \mathrm{Id}$, and hence $h''(D) = h(D)$. Hence, $h''(D)$ is a maximal antichain in P, and we are done.

QED Claim 2.

Since H is P'-generic over N, $N[H]$ satisfies the Truth Lemma. Since $h''(H) \subseteq G$, h has an extension to a map $h^*: N[H] \to \boldsymbol{B}[G]$, given by $h^*(a_H) = (h(a))_G$. Set $\bar{M} = \mathrm{Rg}(h^*)$. Hence, $\bar{M} \prec \bar{\boldsymbol{B}}[G]$, and it is clear that $|\bar{M}| = \alpha^+$. On the other hand, $N[H]$ has the same ordinals as N. Hence, $\mathrm{On} \cap \bar{M} = (h^*)''(\mathrm{On} \cap N[H]) = h''(\mathrm{On} \cap N) = \mathrm{On} \cap M$. So, $\bar{M} \cap \alpha^+ = M \cap \alpha^+$, and $|\bar{M} \cap \alpha^+| \leqq \alpha$. QED Theorem 3.

2.3. We will have to deal with situations where, keeping the notations of the proof of Theorem 3, H cannot be P'-generic over N. Hence, h will not extend to an elementary embedding from $N[H]$ into $\boldsymbol{B}[G]$. Hence, we have got to refine slightly the argument.

Keeping the notations of the proof of Theorem 3, we see that, due to the definition of h^*, $\bar{M} = \{a_G / a \in M\}$. We will keep this definition of $\bar{M}$. However, it will be false that $\bar{M} \cap \mathrm{On} = M \cap \mathrm{On}$ [actually, "$\bar{M} \cap \mathrm{On} = M \cap \mathrm{On}$" is equivalent to "$H$ is P'-generic over N", as shown in [S]]. Hence, we will have to ensure that, although $\bar{M} \cap \alpha^+$ is larger that $M \cap \alpha^+$, we do not add too many ordinals.

2.4. Hence, assume that μ is a regular cardinal, and that $\boldsymbol{B} = (H_\mu, \in, \ldots)$ is a structure, where "..." denote some additional predicates, including a well-order of H_μ. Let P be a set of conditions, such that $\mathscr{P}(P) \subseteq H_\mu$. Let M be an elementary substructure of $\boldsymbol{B}$. Let G be P-generic over V. Set $M[G] = \{a_G / a \in M\}$.

LEMMA 4. $M[G] \prec \boldsymbol{B}[G]$.

PROOF OF LEMMA 4. Let us first observe that the forcing relation over P restricted to $\boldsymbol{B}$ is definable in $\boldsymbol{B}$, since $\mathscr{P}(P) \subseteq H_\mu$. [This point is, however, not essential in the proof, since we could always add it to $\boldsymbol{B}$ as a predicate.]

To show that $M[G] \prec \boldsymbol{B}[G]$, let us take a formula $\varphi(v_1, \ldots, v_n)$ and terms $a_1, \ldots, a_n \in M$ and show that $M[G] \models \varphi((a_1)_G, \ldots, (a_n)_G)$ iff $\boldsymbol{B}[G] \models \varphi((a_1)_G, \ldots, (a_n)_G)$. We proceed as usual, by induction on φ, so that we can assume that φ is of the form $(\exists x)\theta$, and that it is true in $\boldsymbol{B}[G]$. Let $p_0 \in G$ be such that $p_0 \Vdash_P \varphi(a_1, \ldots, a_n)$. Since $\boldsymbol{B} \models \mathrm{ZFC}^-$ and since $\mathscr{P}(P) \subseteq H_\mu$, the maximum principle is valid in $\boldsymbol{B}$. Hence, we can find a term $b \in H_\mu$ such that

$$\boldsymbol{B} \models (\forall q)[q \Vdash_P \varphi(a_1, \ldots, a_n) \rightarrow q \Vdash_P \theta(b, a_1, \ldots, a_n)].$$

By elementarity, such a term b must exists in M. By induction hypothesis, $M[G] \models \theta(b_G, (a_1)_G, \ldots, (a_n)_G)$. QED Lemma 4.

2.5. Remark. We could define $M[G]$ in another (equivalent) way. Let $h : N \underset{\sim}{\rightarrow} M$ be the transitive collapse of M. Let B denote the boolean completion of P. Assume that G is B-generic over V, and set $B' = h^{-1}(B)$ and $H = (h^{-1})''(G)$. Hence, H is an ultrafilter over the N-complete boolean algebra B'. Hence, we can define the boolean model $N^{B'}/H$. It is immediate that h extends to an elementary embedding from $N^{B'}/H$ into $\boldsymbol{B}$, and that the image of this elementary embedding is precisely $M[G]$.

3. Consistency results

3.1. We shall begin with the following theorem.

Theorem 5. *Assume that GCH holds. Assume that there exists an elementary embedding $j : V \rightarrow N$ such that:*

(1) *N is transitive and $j \neq \mathrm{Id}$.*

(2) *If we set $\kappa = \mathrm{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{+\omega+1}} \subseteq N$.*

Then there exists a generic extension of V in which $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$ holds.

Proof of Theorem 5.

Claim 1. In V, $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

Proof of Claim 1. Let L be a countable language and $\boldsymbol{A} = (\lambda^{+\omega+1}, \lambda^{+\omega}, L^{\boldsymbol{A}})$ be a structure. Set $\pi = j \restriction_{\boldsymbol{A}}$. π is an elementary embedding from $\boldsymbol{A}$ into $j(\boldsymbol{A})$. Since $N^{\lambda^{+\omega+1}} \subseteq N$, $\mathrm{Rg}(\pi) = M \in N$ and $M \prec j(\boldsymbol{A})$. It is clear that $|M| = \lambda^{+\omega+1} = j(\kappa^{+\omega+1})$ and that $M \cap j(\lambda^{+\omega}) = j''(\lambda^{+\omega})$, hence that $|M \cap j(\lambda^{+\omega})| = j(\kappa^{+\omega})$. Hence, $N \models$ "$j(\boldsymbol{A})$ admits an elementary submodel of type $(j(\kappa^{+\omega+1}), j(\kappa^{+\omega}))$". By elementarity, $\boldsymbol{A}$ admits an elementary submodel M' of type $(\kappa^{+\omega+1}, \kappa^{+\omega})$. QED Claim 1.

REMARKS. (1) Since $M \cap \lambda = \kappa$ is transitive, we could also assume that $M' \cap \kappa \in \kappa$.

(2) Since $\pi \restriction_\kappa = \mathrm{Id} \restriction_\kappa$, Claim 1 would still be true over languages of arbitrary cardinality $< \kappa$.

We can now forget about the elementary embedding j and use only the fact that, in V, $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$, and some cardinality properties, following from the GCH.

Let P_0 be the set of all finite partial functions $p : \omega \to \kappa^{+\omega}$, ordered by reverse inclusion. Hence, $|P_0| = \kappa^{+\omega}$, and hence in V^{P_0}, $\kappa^{+\omega+1} = \omega_1$. In V^{P_0}, let P_1 be the Levy set making $\lambda = \omega_2$ with countable conditions. Set $P = P_0 * P_1$. It is clear that $|P| = \lambda$, that P has the λ-antichain condition and that, in V^P, $\kappa^{+\omega+1} = \omega_1$, $\lambda = \omega_2$, $\lambda^{+\omega} = \aleph_\omega$ and $\lambda^{+\omega+1} = \aleph_{\omega+1}$. We claim that, in V^P, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$ holds.

To see that, set $\mu = \lambda^{+\omega+1}$, and let G be P-generic over V. Since $\mathscr{P}(P) \subseteq H_\mu$, we see that $(H_\mu)^{V[G]} = H_\mu[G]$. Let $\bar{B} \in V[G]$ be such that $\bar{B} \subseteq H_\mu[G]$ and set $\bar{\boldsymbol{B}} = (H_\mu[G], \in, \bar{B})$. Let $B \in V$ be such that $B \subseteq H_\mu$ and $\bar{B} = (B)_G$. Finally, set $\boldsymbol{B} = (H_\mu, \in, B, \{P\}, \lhd)$, where $\lhd$ is a well-order of H_μ. Let $M \in V$ be such that $M \prec \boldsymbol{B}$, $|M \cap \lambda^{+\omega+1}| = \kappa^{+\omega+1}$ and $|M \cap \lambda^{+\omega}| = \kappa^{+\omega}$. Set $\bar{M} = M[G]$. By Lemma 4, $\bar{M} \prec \boldsymbol{B}[G]$. It is clear that $|\bar{M}| = \kappa^{+\omega+1} = (\omega_1)^{V[G]}$. Hence, it is enough to show that

CLAIM 2. $V[G] \models$ "$|\bar{M} \cap \lambda^{+\omega}| \leqq \omega_0$".

PROOF OF CLAIM 2. Let δ be an ordinal. Say that a is a δ-term if $a \in H_\mu$ and $\Vdash_P$ "$a \in \delta$". Let $T(\delta)$ denote the set of all δ-terms. It is clear that $\bar{M} \cap \lambda^{+\omega} = \{a_G / a \in T(\lambda^{+\omega}) \cap M\}$. Moreover, since $\lambda^{+\omega} = \bigcup \{\lambda^{+n} / n < \omega\}$, and since $(\forall n < \omega)[\lambda^{+n} \in M]$ (because $\omega + 1 \subseteq M$) and $\lambda^{+\omega} \in M$, we see that $\bar{M} \cap \lambda^{+\omega}$ is still equal to $\{a_G /$for some $\delta \in M \cap \lambda^{+\omega}, a \in T(\delta) \cap M\}$. Hence, in order to show that $|\bar{M} \cap \lambda^{+\omega}| \leqq \omega$, it is enough to show that, for all $\delta \in M \cap \lambda^{+\omega}$, $V[G] \models |T(\delta) \cap M| \leqq \omega$. Hence, take $\delta \in M \cap \lambda^{+\omega}$. An element $a \in T(\delta)$ can be obtained as follows. Let B denote the boolean completion of P. Define a function $f : \delta \to B$ setting, for $\alpha < \delta$, $f(\alpha) = \| a = \alpha \|^B$. It is clear that a_G is the unique $\alpha < \delta$ such that $f(\alpha) \in G$. Since P has the λ-antichain condition and $|P| = \lambda$, we see that $|B| = \lambda$ and that $B \in H_\mu$. Hence, $T(\delta) \in H_\mu$ and, moreover, $|T(\delta)| = \lambda^\delta < \lambda^{+\omega}$, since we assume that the GCH holds ($\lambda^{+\omega}$ strong limit would be enough for this point). Since $\boldsymbol{B} \models$ "$|T(\delta)| \leqq \lambda^{+\omega}$", $M \models$ "$|T(\delta)| \leqq \lambda^{+\omega}$". Hence $|T(\delta) \cap M| \leqq \kappa^{+\omega}$. So, $V[G] \models |T(\delta) \cap M| \leqq \omega$.

QED Theorem 5.

3.2. We can improve Theorem 5 as follows.

THEOREM 6. *Assume that the GCH holds. Assume that* $1 \leqq n < \omega$ *and that there exists an elementary embedding* $j: V \to N$ *such that*:

(1) *N is transitive and* $j \neq \mathrm{Id}$.

(2) *Setting* $\kappa = \mathrm{cp}(j)$ *and* $\lambda = j(\kappa)$, $N^{\lambda^{+\omega_n+1}} \subseteq N$.

Then there exists a generic extension of V in which $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$ *holds*.

PROOF OF THEOREM 6. We follow the proof of Theorem 5. As first claim, we obtain the fact that $(\lambda^{+\omega_n+1}, \lambda^{+\omega_n}) \twoheadrightarrow (\kappa^{+\omega_n+1}, \kappa^{+\omega_n})$ holds in V. Moreover, by Remark (2) following the proof of Claim 1, this hypothesis holds over all languages L such that $|L| < \kappa$. In particular, this implies that we can always assume that our elementary substructure M is such that $\omega_n + 1 \subseteq M$.

Now, let P_0 be the set of all partial functions $p: \omega_n \to \kappa^{+\omega_n}$ such that $|P| < \omega_n$, ordered by reverse inclusion. The supplementary point here is that, because of the GCH, $|P_0| = \kappa^{+\omega_n}$. Hence, forcing with P_0 preserves all the cardinals γ such that $\gamma \leqq \omega_n$ or $\gamma > \kappa^{+\omega_n}$, and makes $\kappa^{+\omega_n+1} = \omega_{n+1}$.

We let, in V^{P_0}, P_1 be the Levy collapse making λ the successor of $\kappa^{+\omega_n+1}$ and set $P = P_0 * P_1$. Hence $|P| = \lambda$, P has the λ-antichain condition and, in V^P, $\kappa^{+\omega_n+1} = \omega_{n+1}$, $\lambda = \omega_{n+2}$, $\lambda^{+\omega_n} = \aleph_{\omega_n}$ and $\lambda^{+\omega_n+1} = \aleph_{\omega_n+1}$.

Let us keep the notations of the proof of Theorem 5. Since $\omega_n + 1 \subseteq M$, $\{\lambda^{+i}/i < \omega_n\} \subseteq M$. Hence, $\bar{M} \cap \lambda^{+\omega_n} = \{a_G/\text{for some } \delta \in M \cap \lambda^{+\omega_n}, a \in T(\delta) \cap M\}$. Hence, it is enough to show that, for $\delta \in M \cap \lambda^{+\omega_n}$, $|T(\delta) \cap M| \leqq \kappa^{+\omega_n}$. This is proven exactly as in Theorem 5. QED Theorem 6.

3.3. REMARKS. (1) Of course, the scheme of proof of Theorem 6 applies to a variety of situations, as soon as the right cardinals are preserved.

We did not use fully the full fact that $|P| = \lambda$ and P satisfies the λ-antichain condition. $|P| < \lambda^{+\omega}$ [resp. $|P| < \lambda^{+\omega_n}$] would have been enough.

(2) In general, we could also have argued as follows. Assume the GCH. Let ρ, κ be two regular cardinals, such that $\omega \leqq \rho < \kappa$. Assume the existence of an elementary embedding $j: V \to N$ such that N is transitive, $j \neq \mathrm{Id}$, and, if we set $\kappa = \mathrm{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{+\rho+1}} \subseteq N$. Then, in some generic extension of V, $(\aleph_{\rho+1}, \aleph_\rho) \twoheadrightarrow (\rho^+, \rho)$ holds.

(3) As we have seen in Lemma 1, $(\aleph_{\rho+1}, \aleph_\rho) \twoheadrightarrow (\rho^{++}, \rho^+)$ will, in general, not be consistent.

(4) What about $(\aleph_{\rho+1}, \aleph_\rho) \twoheadrightarrow (\delta^+, \delta)$, with $\delta^+ \leqq \rho$?

More specifically, what about

(a) $(\aleph_{\omega_1+1}, \aleph_{\omega_1}) \twoheadrightarrow (\omega_1, \omega_0)$,
(b) $(\aleph_{\omega_2+1}, \aleph_{\omega_2}) \twoheadrightarrow (\omega_2, \omega_1)$,
(c) $(\aleph_{\omega_2+1}, \aleph_{\omega_2}) \twoheadrightarrow (\omega_1, \omega_0)$?

A glance at the proof of Theorem 6 shows that the construction does not work in this case. For, assume that $0 \leqq k < n < \omega$, that κ, λ are two regular cardinals such that $\omega_n < \kappa < \lambda$ and that $(\lambda^{+\omega_n+1}, \lambda^{+\omega_n}) \twoheadrightarrow (\kappa^{+\omega_n+1}, \kappa^{+\omega_n})$ holds. Assume also that a set of conditions P has been found, such that we can establish that $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{k+1}, \omega_k)$ holds in V^P, using the above hypothesis in V, and the method of proof of Theorem 6. Let G be P-generic over V. In $V[G]$, the following has to hold:
(a) $|\kappa^{+\omega_n}| = \omega_k$,
(b) $\lambda^{+\omega_n} = \aleph_{\omega_n}$.
Since $\omega_k < \omega_n \leqq \kappa^{+\omega_n}$, (a) implies that ω_n is not a cardinal in $V[G]$, contradicting (b).

Hence, we have got to do a slight change in the proof of Theorem 6.

THEOREM 7. *Assume that the GCH holds. Assume that k, n are two integers, such that $0 < k+1 < n$. Assume that there exists an elementary embedding $j: V \to N$ such that:*
(a) *N is transitive and $j \neq$ Id.*
(b) *If we set $\kappa = \mathrm{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{+\lambda+1}} \subseteq N$.*
Then there exists a generic extension of V in which $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{k+1}, \omega_k)$ holds.

PROOF OF THEOREM 7. As in Claim 1 of the proof of Theorem 5, we see that the relation $(\lambda^{+\lambda+1}, \lambda^{+\lambda}) \twoheadrightarrow (\kappa^{+\kappa+1}, \kappa^{+\kappa})$ holds in V. Hence, we shall derive the result from this hypothesis, together with some cardinal assumptions.

Let P_0 be the set of all partial functions $p: \omega_k \to \kappa^{+\kappa}$, such that $|p| < \omega_k$. Using GCH, we see that $|P_0| = \kappa^{+\kappa}$. Hence, forcing with P_0 preserves all cardinals γ such that $\gamma \leqq \omega_k$ or $\gamma \geqq \kappa^{+\kappa+1}$. Moreover, $\Vdash_{P_0}$ "$\omega_{k+1} = \kappa^{+\kappa+1}$".

In V_{P_0}, let P_1 be the Levy set of conditions making $\lambda = \omega_n$. Forcing with P_1 over V^{P_0} does not destroy $\kappa^{+\kappa+1}$, precisely because $k+1 < n$. Set $P = P_0 * P_1$. Hence $|P| = \lambda$, P has the λ-antichain condition and in V^P, $\kappa^{+\kappa+1} = \omega_{k+1}$, $\lambda = \omega_n$, $\lambda^{+\lambda} = \aleph_{\omega_n}$ and $\lambda^{+\lambda+1} = \aleph_{\omega_n+1}$. As usual, set $\mu = \lambda^{+\lambda+1}$ and let $B \subseteq H_\mu$. Set $\boldsymbol{B} = (H_\mu, \in, B, \{P\}, \lhd)$, where $\lhd$ is a well-order of H_μ. Let G be P-generic over V, and set $\bar{\boldsymbol{B}} = \boldsymbol{B}[G]$. We have got to find some $\bar{M} \in V[G]$ such that $\bar{M} \prec \bar{\boldsymbol{B}}$, $|\bar{M}| = \omega_{k+1}$ and $|\bar{M} \cap \lambda^{+\lambda}| = \omega_k$. As before, we take a set $M \in V$

such that $M \prec \boldsymbol{B}$, $|M| = \kappa^{+\kappa+1}$ and $|M \cap \lambda^{+\lambda}| = \kappa^{+\kappa}$, and we set $\bar{M} = M[G]$. It is clear that $|\bar{M}| = \kappa^{+\kappa+1} = \omega_{k+1}$. Hence, we have got to show that, in $V[G]$, $|\bar{M} \cap \lambda^{+\lambda}| \leqq \omega_k$.

To show this, we shall anew "count the $\lambda^{+\lambda}$-terms".

Assume that $\xi \in \bar{M} \cap \lambda^{+\lambda}$. Let $a \in M$ be such that $\xi = a_G$. Applying the maximum principle in H_μ, we see that there exists a term $b \in H_\mu$ such that

(1) $\Vdash_P b < \lambda^{+\lambda}$,

(2) $\Vdash_P a < \lambda^{+\lambda} \rightarrow a = b$.

[In particular, for any such b, $b_G = a_G = \xi$.]

By elementarity, such a b must exist in M. This means that we can, without loss of generality, assume that

(a) $\Vdash_P a < \lambda^{+\lambda}$.

Now, we shall use the fact that P satisfies the λ-antichain condition. This implies that

(b) $\boldsymbol{B} \models$ "for some $\delta < \lambda^{+\lambda}$, $\Vdash_P a < \delta$".

By elementarity, there exists some $\delta \in M \cap \lambda^{+\lambda}$ such that $\Vdash_P a < \delta$, which means that $a \in T(\delta)$ [the set of all δ-terms]. Hence, we have shown that

(c) $\bar{M} \cap \lambda^{+\lambda} = \{a_G/\text{for some } \delta \in M \cap \lambda^{+\lambda}, a \in T(\delta) \cap M\}$.

This implies that $|\bar{M} \cap \lambda^{+\lambda}| \leqq |M \cap \lambda^{+\lambda}| \cdot \sup\{|T(\delta)|/\delta \in M \cap \lambda^{+\lambda}\}$. Hence, let us verify that, in $V[G]$, this cardinal is at most ω_k.

(i) $V[G] \models |M \cap \lambda^{+\lambda}| = |\kappa^{+\kappa}| = \omega_k$.

(ii) Assume that $\delta \in M \cap \lambda^{+\lambda}$, and let us show that $V[G] \models |T(\delta)| \leqq \omega_k$. Let B denote the boolean completion of P. Since $|P| = \lambda$ and P satisfies the λ-antichain condition, we see that $|B| = \lambda$. Hence, as in the proof of Theorem 5, we see that $\boldsymbol{B} \models |T(\delta)| \leqq |B^\delta| = \lambda^\delta < \lambda^{+\lambda}$, since $\delta < \lambda^{+\lambda}$ and the GCH holds in V. Hence $M \models |T(\delta)| \leqq \lambda^{+\lambda}$, and so $V[G] \models |T(\delta) \cap M| \leqq |\kappa^{+\kappa}| \leqq \omega_k$. QED Theorem 7.

3.4. Remarks. Hence, we have solved the cases $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{k+1}, \omega_k)$, except one: $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_n, \omega_{n-1})$, with $n \geqq 1$. We shall leave this case open.

References

[DJK] H.-D. Donder, R. Jensen and B. Koppelberg, *Some applications of the core model*, in *Set Theory and Model Theory*, Lecture Notes in Math. No. 872, Springer-Verlag, Berlin, 1981, pp. 55–97.

[DK] H.-D. Donder and P. Koepke. *On the consistency strength of "accessible" Jonsson cardinals and of the weak Chang Conjecture*, Ann. Pure Appl. Logic **25** (1983), 233–261.

[DKL] H.-D. Donder, P. Koepke and J.-P. Levinski, *Some stationary subsets of* $\mathscr{P}(\lambda)$, Proc. Am. Math. Soc. **102** (1988), 1000–1004.

[DL] H.-D. Donder and J.-P. Levinski, *Some principles related to Chang's Conjecture*, Ann. Pure Appl. Logic, to appear.

[F] M. Foreman, *Large cardinals and strong model-theoretic transfer properties*, Trans. Am. Math. Soc. **272** (1982), 427–463.

[K] K. Kunen, *Saturated ideals*, J. Symb. Logic **43** (1978), 65–76.

[L] J.-P. Levinski, *Instances of the Conjecture of Chang*, Isr. J. Math. **48** (1984), 225–243.

[R] F. Rowbottom, *Some strong axioms of infinity incompatible with the axiom of constructibility*, Ann. Math. Logic **3** (1971), 1–44.

[S] S. Shelah, *Proper Forcing*, Lecture Notes in Math. No. 940, Springer-Verlag, Berlin, 1982.